Les lais: or, what ever became of Mesopotamian Mathematics?

The obvious

When asking whether Mesopotamian mathematics left anything to later mathematical cultures, an obvious answer is "Yes – at least rudiments of the sexagesimal place value system" – namely the minutes and seconds of our time-keeping and our division of the degree. To Greco-Roman Antiquity and to the Islamic and Christian Middle Ages it left more than rudiments, namely the systematic use of sexagesimal place value *fractions* – a striking way to change the original floating-point system into a fixed-point-system, since the integer part of numbers were not written sexagesimally.

Together with the notation came the *idea* of place-value, and thus ultimately our own use of decimal fractions. Al-Uqlīdisī's proposal to use decimal fractions [Saidan 1978: 481*f*] as well as Jordanus de Nemore's generalization to any base (called by him "consimilar fractions" [Eneström 1913: 43]) were derived from the sexagesimal fractions with which they were familiar.

The sexagesimal fractions were mainly used in astronomy, and were indeed also a legacy of Babylonian mathematical astronomy. One may ask whether this part of Babylonian science left more of its mathematics.[1] As a matter of fact it did, some of the arithmetical methods were taken over by Greek astronomy [Jones 1993] as well as Hellenistic astrology [Neugebauer 1988]; but this transmission did not spread to other environments, not even to the "philosophical" astrology of a Ptolemy (although aspects of Mesopotamian astrology did), and its very existence has indeed only been discovered recently.

Contested conventional wisdom

Shortly after the identification of apparently algebraic cuneiform texts of Old Babylonian as well as Seleucid date, Otto Neugebauer suggested that even this level of Babylonian mathematics had influenced Greek and hence later mathematics. In his argument entered the assumption, current at the time, that Greek mathematics had undergone a "foundation crisis" at the discovery of

[1] That it left astronomical and astrological knowledge (or "astrological persuasions", if one prefers) is a familiar matter but not my topic; nor is the legacy of astronomical knowledge to India [Pingree 1978: 536–554].

incommensurability [Hasse & Scholz 1928], similar to that *Grundlagenkrise* which had recently made itself felt in German mathematics. He noticed that the Old Babylonian procedures (which he took to be purely numerical) were structurally similar to the proofs of the so-called "geometric algebra" of *Elements* II and concluded [1936: 250]:

> Die Antwort auf [...] die Frage nach der geschichtlichen Ursache der Grundaufgabe der gesamten geometrischen Algebra,[2] kann man heute vollständig geben: sie liegt einerseits in der aus der Entwicklung der irrationalen Größen folgenden Forderung der Griechen, der Mathematik ihre Allgemeingültigkeit zu sichern durch Übergang vom Bereich der rationalen Zahlen zum Bereich der allgemeinen Größenverhältnisse, andererseits in der daraus resultierenden Notwendigkeit, *auch die Ergebnisse der vorgriechischen "algebraischen" Algebra zu übersetzen*.
>
> Hat man das Problem in dieser Weise formuliert, so ist alles Weitere vollständig trivial und liefert *den glatten Anschluß der babylonischen Algebra an die Formulierungen bei Euklid*.

This translation thesis was accepted widely, also by scholars who knew too little about Babylonian and Greek mathematics to be able to evaluate it; putting it sharply we may say that for most it became a piece of conventional wisdom. Not for Neugebauer, of course, who knew what he was speaking about (and knew much more than he put into writing[3]); in [1963: 530] he added that the Babylonian heritage had become "common mathematical knowledge all over the ancient Near East", and that a (historically rather implausible) direct translation of cuneiform tablets hence needed not be involved.

Neugebauer's thesis was only attacked more than three decades after it was presented, and actually at the level of conventional wisdom. The main contesters were Arpád Szabó [1969] and Sabetai Unguru [1975], both much more familiar with the Greek than with the Babylonian material. None of them had noticed what Neugebauer stated in 1963, and both made much of the incompatibility between the arithmetical and the geometrical approach (Unguru also of the supposed incompatibility between the solution of a problem and the justification

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[2] Namely, the application of an area with deficiency or excess, *Elements* II.5–6; this gives rise to the characteristic use of semi-sum and semi-difference of the sides of a rectangle – in other words, their average and the deviation.

[3] Cf. [Høyrup 2002a: 274 and *passim*]. Indeed, as Neugebauer stated explicitly concerning his monumental *Mathematische Keilschrift-Texte*, it did no belong "zu den Aufgaben, die ich mir in dieser Edition gestellt habe, die Konsequenzen zu entwickeln, die sich nun aus diesem Textmaterial ziehen lassen" [Neugebauer 1935: III, 79]. Being first of all a historian of astronomy, he never took the time to draw these consequences in writing, even though he published another volume of texts together with Abraham Sachs [1945].

of the methods used to solve it).[4] At least Unguru's attack aroused a certain echo, being published in English in a major journal, being held in an unusually aggressive tone, and receiving equally vicious replies from André Weil [1978] and Hans Freudenthal [1977] (and a gentle response from van der Waerden [1976]). On the whole, however, the general belief in a translation of Babylonian numerical into Greek geometrical "algebra" survived; the survival was facilitated by lacking recognition of certain fundamental differences between Old Babylonian and Seleucid procedures (at the moment no intermediate "algebraic" texts were known); the Euclidean "application of areas" is indeed related to the Old Babylonian but not to Seleucid procedures.

Neugebauer vindicated – with a twist

From 1982 onward, I succeeded gradually in convincing that part of the scholarly world which was interested in the matter that the numerical interpretation of the Babylonian "algebraic" texts is mistaken: the majority of its problems really deal with those measurable geometric entities of which they speak, and this geometry of measurable lines and areas constitutes the basic representation by means of which other problems (about mutually reciprocal number pairs, about commercial rates, etc.) are solved – just as numbers constitute the basic representation for *our* solution of problems about measurable entities of any ontological kind.[5] The method, moreover, was *analytic*, that is, it treated the representatives of the unknown quantities as known quantities would have been treated (exactly as we do with our x and y), and it was reasoned to the same extent (and in much the same sense) as our solution of equations. Babylonian algebra was thus, to a far larger extent than evident in the reading of the texts as numerical algorithms, a real *algebra*. By being geometric it was also even more similar to the geometry of *Elements* II than Neugebauer had supposed.

However, the same analysis revealed much more. Firstly of all, it highlighted

[4] For some reason, none of them took notice of the suspicious similarity between part of Diophantos's indubitably numerical *Arithmetica* (I: 27–28, 30) and the critical theorems of Elements II.

[5] I shall not dwell on the arguments for this – they are set forth in depth and detail in [Høyrup 2002a] together with arguments for what else follows in this section of the article. The first suggestion of my thesis was presented (in Danish) to the 1982-meeting of the Danish National Committee for the History and Philosophy of Science.

the difference between Old Babylonian and Seleucid texts and showed that part of what was just said about the general character of "Babylonian algebra" and its similarity with *Elements* II is only true of its Old Babylonian phase. The few Late Babylonian "algebraic" texts (the Seleucid specimens known since long, and the Late Babylonian but pre-Seleucid text W 23291 analyzed in [Friberg 1997]) differ from the Old Babylonian ones in several respects. Firstly, only the simplest "elements" of the old discipline – problems about rectangles with no arbitrary coefficients – turn up in the late phase; they do not serve in representation and thus do not constitute an algebra in any proper sense. Moreover, while W 23291 still uses the technique of average and deviation, the Seleucid method builds on sum and difference;[6] the Seleucid texts also contain new problem types involving the diagonal of rectangles (for instance, to find the sides from the area and the sum of the sides and the diagonal).

Some of the Sumerograms found in the Late Babylonian texts turn out to be *new* translations from Akkadian (or perhaps Aramaic); this higher level of mathematics thus cannot have been transmitted directly within the environment of scholar-scribes from Old to Late Babylonian times. On the other hand, the Seleucid reappearance of problems where the sum of reciprocal numbers is given (a familiar Old Babylonian type) shows that at least one transmission channel was familiar with the sexagesimal system – but since precisely these problems still make use of the average-deviation procedure, this channel is likely not to be the only one. As in the case of omen science, we may even imagine that part of the transmission has passed through Elamite, Hittite or other peripheral areas.

Close scrutiny of the Old Babylonian mathematical terminology[7] is also

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[6] If no diagrams are to be used, the difference between the two approaches is most easily explained in symbolic algebra. We may consider a rectangle ▭(*x*,*y*) whose area *A* is known together with the difference *d* between the sides – arithmetically hence

$$xy = A \;,\;\; x-y = d \;.$$

The Old Babylonian and Euclidean construction corresponds to the calculations

$$\left(\tfrac{x+y}{2}\right)^2 = \left(\tfrac{x-y}{2}\right)^2 + xy \;\;,\;\; \tfrac{x+y}{2} = \sqrt{\left(\tfrac{d}{2}\right)^2 + A} \;\;,\;\; x = \tfrac{x+y}{2} + \tfrac{x-y}{2} \;\;,\;\; y = \tfrac{x+y}{2} - \tfrac{x-y}{2}$$

the Seleucid procedure to

$$(x+y)^2 = (x-y)^2 + 4xy \;\;,\;\; x+y = \sqrt{d^2 + 4A} \;\;,\;\; x = \tfrac{1}{2}([x+y] + [x-y]) \;\;,\;\; y = [x+y] - x \;\;.$$

[7] See [Høyrup 2000] or (with added information about early Old Babylonian Ur and Nippur) [Høyrup 2002a: 317–361]. I use the opportunity to correct a mistake in the latter publication (p. 354): the tablets CBS 43, CBS 154+921 and CBS 165 are *not* from Nippur

informative about transmission channels. Firstly, it reveals a split between the former Ur III core area and those peripheral areas that were only submitted to Ur III between c. 1975 and 1925 (Susa, and the northern-central area with Sippar and Ešnunna). In particular, only texts from the periphery announce results by the phrase "you see" – mostly in Akkadian, *tammar*, but at times with a Sumerian igi (often in non-standard orthography);[8] even Sargonic school texts "see" results – but they use pád. Even here, the transmission thus cannot have been carried by scribes trained in Sumerian.

Investigation at large of the Old Babylonian terminology shows that exactly the metalanguage – that lexicon which is needed not to speak about the operations that are performed but in order to formulate problems and to structure the description of the procedure – is almost fully devoid of Sumerographic writings in the early Old Babylonian texts (later, Sumerograms and pseudo-Sumerograms turn up). The whole Ur III school appears to have trained only that level of mathematics which served directly in its accounting[9] and to have avoided the use of problems.[10]

All this flows together as evidence that even between Sargonic and Old Babylonian times, the cluster of geometric problems that was to unfold as Old Babylonian algebra was carried by a non-scribal, Akkadian-speaking environment – no doubt an environment of surveyors. Ultimately, of course, the methods of practical surveying go back to the "learned" administrators of the proto-literate phase.[11] Whether the "surveyor" mentioned in house-sale

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and were not claimed by Eleanor Robson to be so; indeed, as she tells me (personal communication, 28.2.2002), they were bought at the antiquity market before the Nippur excavation started.

8 Indirect references to the idiom of "seeing" and a few slips in copies show that the phrase was known in the core area but deliberately avoided.

9 Evidently this fits well king Šulgi's unusual modesty when he speaks in "Hymn B" about his knowledge of mathematics: addition, subtraction, counting and accounting – ed. [Castellino 1972: 32] (Castellino's translation misunderstands the text at this point). Evidently, Šulgi's ghostwriter knew of no other mathematics.

10 This question is treated in depth in [Høyrup 2002b].

11 Indeed, the redefinition of what had once been "natural" (irrigation, ploughing or seed) measures [Powell 1972: *passim*] as units defined in terms of the length unit nindan is attested in Uruk IV [Damerow & Englund 1987] and not likely to antedate writing. Without this redefined metrology, areas of rectangles and right triangles could not be determined from their sides.

contracts from Šuruppak was a specialized scribe or a non-scribal practitioner we do not know.[12] The appearance of rectangle and square problems (not yet of the second degree but only asking to find one rectangle side from the area and the other side and a square side from the area) precisely in the Old Akkadian school could be due to their presence in an akkadian-speaking environment, but this conclusion is certainly not mandatory. However, the reappearance of the Sargonic pád as an Old Babylonian igi cannot be explained without a transmission in non-Sumerian language (whence, by necessity, in Akkadian).

It is also next to certain that the "quadratic completion" (the trick which is needed to solve the second-degree problems) was invented in this Akkadian-speaking environment somewhere between the end of the Sargonic epoch (in whose schools it left no trace) and early Old Babylonian times. And indeed, a didactical text from Susa which explains it refers to it as "the Akkadian [method]".[13]

One of the characteristics of oral culture is its eristic orientation; its riddles are not meant as entertainment but as challenges.[14] This also holds for pre-Modern non-scribal mathematical professions, whether accountants or surveyors. In many cases where their knowledge was adopted by literate environments and thus brought into writing, the problems that are taken over are introduced by phrases like "If you are an accomplished calculator, tell me ..."; that is, in their original contexts these problems were *riddles for professionals*, and only those belonging to the profession knew to answer them correctly. Being *for professionals*, they had to be concerned with such things as fell under the responsibility of the profession. A knight might show his valour not only in real war but also in the fictive aestheticized war of the tournament; a surveyor had to show *his* by being able to solve difficult problems about the measures of fields – perhaps fairy-tale problems never encountered in real life (like knowing the sum of the four sides and the area of a square field but not its side), in any case problems whose solution asked for virtuosity beyond trite routine.

For such purposes, riddles like these were perfect:

– On a single square with side s and area $\square(s)$ ($_4s$ stands for "*the* four sides",

[12] [Krecher 1973: 172–176]. He is spoken of as the um.mi, "master", who applied the measuring rope; that um.mi was borrowed into Akkadian as *ummânum* – an expert artisan rather than a scholar – supports the non-scribal interpretation but does not prove it.

[13] TMS IV, cf. [Høyrup 2002a: 90–94].

[14] The classical discussion of this is [Ong 1967], but see also [Pucci 1996]. For the application to mathematical practitioners, cf. [Høyrup 1990a].

Greek letters for given numbers):

$$\square(s) = \alpha \qquad \text{(a)}$$
$$s+\square(s) = \alpha \qquad \text{(b)}$$
$${}_{4}s+\square(s) = \alpha \qquad \text{(c)}$$
$$\square(s)-s = \alpha \qquad \text{(d)}$$
$$s-\square(s) = \alpha \qquad \text{(e)}$$

– On two concentric squares with sides s_1 and s_2:

$$\square(s_1)+\square(s_2) = \alpha\ ,\ \ s_1 \pm s_2 = \beta \qquad \text{(f)}$$
$$\square(s_1)-\square(s_2) = \alpha\ ,\ \ s_1 \pm s_2 = \beta \qquad \text{(g)}$$

– On a circle with circumference c, diameter d, and area A:

$$c+d+A = \alpha \qquad \text{(h)}$$

– On a rectangle with sides l and w and diagonal d:

$$\sqsubset\!\sqsupset(l,w) = \alpha\ ,\ \ l = \beta \text{ or } w = \gamma \qquad \text{(i)}$$
$$\sqsubset\!\sqsupset(l,w) = \alpha\ ,\ \ l \pm w = \beta \qquad \text{(j)}$$
$$\sqsubset\!\sqsupset(l,w)+(l \pm w) = \alpha\ ,\ \ l \mp w = \beta \qquad \text{(k)}$$
$$\sqsubset\!\sqsupset(l,w) = \alpha\ ,\ \ d = \beta \qquad \text{(l)}$$
$$\sqsubset\!\sqsupset(l,w) = l+w \qquad \text{(m)}$$

With a small proviso for (e), all of these (and probably no others except perhaps the square problem $d-s = 4$ with the mock solution $s = 10$) appear to have circulated among the lay surveyors when the Old Babylonian scribe school borrowed from them and developed its fabulous algebraic discipline. As we notice, there are no arbitrary coefficients, and no representation – everything really deals with square and rectangular fields.

In W 23891, we encounter (a), (i) and (j). Together with new rectangle riddles involving the diagonal, (j) is found in a Seleucid text with the new method.[15] (b), (d), (f), (g) and (j) all turn up (not as problems but as justifications of the way they were solved in pre-Seleucid times) in *Elements* II, (f), (g) and (j) (as number problems) in Diophantos's *Arithmetica* I. A Demotic papyrus[16] contains (l) with "Seleucid" solution, together with a sequence of problems about a pole leaned against a wall, some in a simple variant involving only the Pythagorean theorem and known already from an Old Babylonian text, others in a

[15] We should remember that we have essentially one relevant Late Babylonian, pre-Seleucid text and one Seleucid text (BM 34568). When not copied by scholar-scribes, the mathematical texts from this period were probably written on wax tablets (most likely in Aramaic); one of the scholarly copies tells indeed to be made from a wax original.

[16] P. Cairo *J.E.89127–30,89137–43*, ed. [Parker 1972], from the third century BCE.

sophisticated variant known from the "Seleucid" text BM 34568. Another Demotic Papyrus[17] contains a couple of summations of series "from 1 to 10", obviously related to two other summations found in the Seleucid text AO 6484. The Seleucid as well as the Demotic texts are thus certainly evidence of "common mathematical knowledge all over the ancient Near East", but whether the innovations were made in Syria, in Egypt or in Mesopotamia (or even further east – the summation formulae have unmistakable though later Indian kin) we cannot know.

In any case, (c), (h) and (l) (the latter in "Seleucid-Demotic" shape) turn up in pseudo-Heronian and Latin agrimensorial material. (b), (c) and (e) were used by al-Khwārizmī in the ninth century CE for his geometric proofs of the solution to mixed quadratic equations; since (c) is less adequate than (b) for the formula he wants to justify, it must have been familiar either to himself or to his public. Almost everything (except (f) and (g), but together with the Seleucid rectangle-diagonal problems) is also found in Arabic treatises about mensuration (including Savasorda's treatise, which presents Arabic knowledge in Hebrew), and (c), (j) and (k) turn up in Italian *abbaco* sources in a way that cannot be derived from known Latin translations from the Arabic.[18]

In the main, Neugebauer was thus fully right; but what went into Greek "geometrical algebra" was not the sophisticated scribal algebra of the Old Babylonian era but the lay and anonymous riddle tradition – even this of Mesopotamian origin but by the later first millennium common knowledge in the whole area where Assyrian and Persian armies and army surveyors and, probably more important, the Assyrian and Persian administrators had passed.

The problems also reached India. Embedded in the geometrical section of Mahāvīra's ninth-century *Gaṇita-sāra-saṅgraha* we find problems of indubitably Old Babylonia origin – for instance (h) – together with a way to find inner heights in an arbitrary triangle which almost certainly arose in the Near-Eastern surveyors' environment in Late Babylonian but pre-Seleucid times,[19] and a number of the characteristic Demotic-Seleucid rectangle problems; strikingly, these borrowings are located in different chapters in accordance with their age,

17 P. British Museum *10520*, ed. [Parker 1972], probably of early Roman date.

18 Leonardo Fibonacci also has a number of the problems, but mainly copied from Gherardo da Cremona's translation of Abū Bakr's *Liber mensurationum*.

19 Even this method (reshaped and generalized in *Elements* II.12–13 so as to hold even for outer heights in obtuse-angled triangles) returns in pseudo-Heronian and Medieval Arabic treatises – see [Høyrup 1997]. It is impossible to know whether it was strictly of Mesopotamian origin.

as if the import had come in three separate waves.[20]

Mahāvīra was a member of the Jaina community. Nothing similar is found in non-Jaina writings I know about. The direct influence of Mesopotamian astronomical methods in India thus has no counterpart within mathematics in general (apart from what was said above concerning the summation of series, where a link is certain but the direction of influence undetermined).

Simpler matters and phraseology

If both an Old Babylonian tablet (BM 13901 #23) and Abū Bakr's *Liber mensurationum* [ed. Busard 1968: 87] tells that "I have aggregated the four sides and the area" (namely of a square, in both cases mentioned previously); if in both cases the appearance of the sides before the area is unexpected, given the habits of the times; and if the resulting side is 10 in both cases; then the existence of a shared tradition is not subject to reasonable doubt.

On the other hand, the fact that two mathematical cultures share the "surveyors' formula" for the area of an approximately rectangular quadrangle (average length times average width) proves nothing as to their being connected – even though the formula is only approximate, the idea is simply too close at hand once rectangular areas are determined as the product of length and width. Similarly, taking the perimeter of a circle to be thrice the diameter is an adequate approximation for many purposes, and its being shared has no certain implications. Even though much from the basic level of Mesopotamian practical mathematics reappears in pseudo-Heronian writings from the Hellenistic age, it is therefore not *prima facie* obvious that such simple matters were borrowed.

However, even simple mathematics goes together with language, and at times indubitable traces of borrowing survives the translation involved; several instances of shared phrases point to the existence of shared mathematical traditions with a Mesopotamian core and reaching beyond the surveyors' culture.

One such phrase has to do exactly with the perimeter of the circle (and thus remains within the field of practical geometry). It is known but not much noticed that the Old Babylonian operation by which the perimeter is determined from the diameter is to "triple" it (*šalāšum*, e.g., BM 85194 obv. I.47) or to "repeat in 3 steps"(a.rá 3 tab.ba, obv. II.44,50); it is never a multiplication by 3, expressed by the verb *našûm*/íl, the operation invariably used when reciprocals, igi.gub-

[20] Details and documentation can be found in [Høyrup 2004].

factors[21] or any other operation of proportionality is involved, nor certainly the construction of a rectangular area (*šutakūlum*, with a wealth of synonyms and logograms), used also when 12 times the circular area is found as the square on the perimeter.

This peculiarity returns not only in the pseudo-Heronian treatise but also when Hero refers to the habits of practitioners. Invariably, the expressions τρισσάκις and τριπλάσιον are used, even when neighbouring multiplications are ἐπὶ *n*.

A fifteenth-century German source shows us that this is no linguistic quirk but a reflection of a practice, which is obviously the reason that it has survived the translation. In Mathes Roriczer's *Geometria deutsch* [ed. Shelby 1977: 121] we find the following prescription:

> If anyone wishes to make a circular line straight, so that the straight line and the circular are the same length, then make three circles next to one another, and divide the first circle into seven equal parts

one of which is marked out in continuation of the three circles. The addition of a seventh is obviously a post-Archimedean innovation[22] and irrelevant in the present context; but the whole explanation shows that the length of the perimeter followed from a construction and that it was *measured* without calculation. This procedure must already have been used in Old Babylonian times and be the reason that the mathematical texts speak as they do.[23]

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[21] Including of course the multiplication of the circular perimeter by the factor 0;20, which produces the diameter.

[22] Already Hero distinguishes (*Metrica* I.xxx–xxxi, ed. [Schöne 1903: 72–74]) between "those who took the perimeter to encompass the triple of the diameter" and those according to whom the perimeter is "the triple diameter and in addition $\frac{1}{7}$ of the diameter"; Hero himself multiplies by 22 and divides by 7 [ed. Schöne 1903: 66].

[23] A somewhat similar case may be constituted by the medieval conservation of the distinctions between the two different additive operations *waṣābum* and *kamārum* and between the subtractions "by removal" (*nasāḫum*) and "by comparison" (*eli ... watārum*), for instance in Abū Bakr's *Liber mensurationum*. From Gherardo's translation of Abū Bakr's text itself it is not obvious that the geometric procedures were still in use, but Jean de Murs' treatment of problem (k) in *De arte mensurandi* [ed. Busard 1998:] shows that he knew them – supporting himself on a diagram and taking advantage of the fact that l–w = 2 he reduces the problem to (c) (which he does not treat, which shows that he must be copying); cf. [Høyrup 1999].

Within this geometric representation, the distinctions are meaningful, and it would be almost impossible to lose them; if everything had been thought of in numerical terms, they would probably have been blurred.

Two other phrases that are common in pseudo-Heronian and certain medieval Arabic writings point back to interaction between lay and scribal mathematics in the Old Babylonian age. One occurs, among numerous other places, in the pseudo-Heronian versions of (c) and (h) [ed. Heiberg 1912: 418, 444] and Leonardo Fibonacci's version of (c) [ed. Boncompagni 1862: 59], which require that the four sides and the square area respectively the circular perimeter, diameter and area be "separated". It is also found (as *berûm*) in the Old Babylonian text (AO 8862, IV.21 [Neugebauer 1935: I, 112]), an algebraic problem about the sum of men, days and the bricks they produce (the number of these being proportional to the number of man-days); in BM 10822 §1, where three types of bricks are to be singled out [Friberg 2001: 90]; and in a slightly different function in TMS VII A, 4, a didactical text about how to treat an indeterminate first-degree equations (thus far removed from the surveyors tradition) [Høyrup 2002a: 182]. Although all are "algebraic", they are either close to scribal computation or meant to point in their formulation toward that area, and we must therefore presume that the idea of separating a sum into constituents (the inverse operation of the symmetric operation *kamārum*/UL.GAR/g̃ar.g̃ar) comes from here rather than from the lay surveyors.

The case of *kayyamānum*/καθολικῶς/ἀεί/*semper* is similar. In pseudo-Heronian and medieval writings, these terms are used to indicate that a particular numerical step in a procedure is made independently of the numerical parameters involved – for instance, in (c), that a halving of 4 is independent of the value of α. It occurs in two mathematical texts from Susa, TMS XIV and TMS XII. Its use in TMS XIV is unclear, but in any case the problem deals with a grain pile [Robson 1999: 119–122]. TMS XII is an "algebraic" but quartic problem [Muroi 2001], thus certainly beyond the horizon of the surveyors, and here the term appears to serve as in the later sources.

Other kinds of evidence confirm that Mesopotamian Bronze Age mathematics participated in what was at least later to develop into a transcultural network (if not community) of mathematical practitioners; for instance, the first known appearance of the problem of continued doublings is in a text from Old Babylonian Mari [ed. Soubeyran 1984: 30]. The details of the formulation and the fact that the doublings are 30 in number shows beyond doubt that this occurrence is related to those of Greek Antiquity and the Middle Ages [Høyrup 1990a: 74] – but not that the problem originated in Mesopotamia. In later times, the network was probably carried by trade connections, and it only happens to become visible in places where it collided with literate mathematical cultures leaving surviving sources. A similar transcultural practitioners' network, if it existed in the beginning of the second millennium, had no chances to become visible to us except through its contacts with Mesopotamian and Egyptian scribes. We know, however, that Mesopotamia was involved in trade with quite remote regions already in the Bronze Age, and therefore cannot exclude that the network *was* transcultural already in the eighteenth century BCE.

Once the existence of this network is established, a number of other terminological similarities between Babylonian and later mathematical texts can be supposed with increased certainty to be the result of loan-translations – thus the Greek notion that a rectangle is "contained" by two sides (περιέχο, corresponding to *šutakūlum*); the reference to a square figure "being" its side and "having" an area as δύναμις, corresponding to *mitḫartum* [Høyrup 1990b]; and the rampant use of "positing" (*jaʿala*, *ponere*, *porre*) in medieval texts corresponding to Old Babylonian *šakānum*/g̃ar.

The least obvious: metamathematics

Numbers, formulae, procedures, terminology: all of this may be said to belong to mathematics proper. The way mathematics was thought about and the way it influenced thinking about other matters, this may (with a slight broadening of the normal use of the term) be spoken of as "metamathematics". Did Mesopotamian metamathematics leave traces in later times?

It is certainly easy to find parallels, and I shall discuss two of them. But the existence of parallels implies neither inspiration nor continuity.

One parallel is inherent in the very notion of "mathematics". In discussions about so-called "ethnomathematics" the point has been made that "mathematics" considered as a closed and coherent field is *our* concept. However, if we consider Old Babylonian problem texts containing several problems we find that these may restrict themselves to a particular "theme" (a rectangular excavation, "algebraic" problems about squares, problems about subdivisions of squares); they may also be "anthologies" and mix mathematical problems of different kinds; but they never mix mathematics with non-mathematical topics (not even with numerology); what we collect under the heading of "Old Babylonian mathematics" was thus also a closed entity in the view of its own times.[24]

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[24] This conclusion is not affected by the fact that student's training pads from the more elementary level of school regularly contain a Sumerian proverb on the obverse and a numerical calculation on the reverse; this just means that the student had to train both things on the same day, as confirmed by the edifying poem known as "Schooldays" [ed. Kramer 1949: 201, 205, cf. correction p. 214].

It appears, however, that this cognitive autonomy of mathematics did not survive the transformations of scribal culture following upon the Kassite conquest (and the disappearance of scribal mathematics from the archaeological horizon for more than a millennium). Indeed, W 23273, a metrological table of Late Babylonian but pre-Seleucid date, starts by listing the sacred numbers of the gods – see [Friberg 1993: 400]. Late Babylonian cuneiform mathematics, as we know, was written by scribes identifying themselves as exorcists, scribes of *Enūma Anu Enlil*, etc. This professional identity did not prevent the astronomer-astrologer-scribes of the Seleucid period from keeping their astronomical tables apart from what we might term the "occult" aspect of their activity; but non-astronomical mathematics was probably so peripheral for them that it did not present itself as something worth to be kept distinct. In any case, there is no evidence that the ancient Greek notion of mathematics as an autonomous field[25] was inherited from the Old Babylonian scribes via the later scribal tradition. It certainly cannot be fully excluded that non-scribal mathematical practice played a role here, as in the transmission of mathematics itself – but as long as no single piece of evidence speaks in favour of such a hypothesis, it remains gratuitous, not least because of the attitude of Greek "theoreticians" to practitioners, similar to that of the classical South-State gentleman who might well spend the darkness of the night with a slave woman but would never be seen with her at breakfast. In all probability, Greek "mathematics", to the extent it was at all thought about as one thing, was a Greek reinvention.

The other parallel has to do with the legitimation of statal power and the way the tasks of the state were understood.[26] It is well known that the emergence of the Mesopotamian state is intimately linked to the invention of writing in the later fourth millennium, and that the purpose of the invention of writing was book-keeping, that is, the application of mathematics.[27] Already the accounts of the proto-literate period are organized in a way which feels familiar, with single contributions and total. This "Cartesian product" was also transferred to the way the social structure was conceptualized in the "List of

[25] Indeed a notion belonging to only a small minority of literate Greeks if we consider the long run of Antiquity. Neo-Pythagoreans were far more numerous than "mathematicians", and even though a Nicomachos was still able to keep arithmetic distinct from numerology, those who read him were mostly not.

[26] The following reflections are extremely sketchy; some substance is provided in [Høyrup 1994: 45–87, 296–306].

[27] See, e.g., [Nissen, Damerow & Englund 1993].

Professions". The introduction of this extensive accounting appears to be linked to the stepwise transformation of an original redistributive economic structure at village level to the bureaucratic structure of Uruk IV – rations of grain etc. were distributed to workers in fixed rations which it was the task of the accounting to control, and land was distributed to high officials in mathematically determined ratios. Although the documents of the time do not speak directly about such things, the state structure was apparently legitimized as securing "just measure". In Ur III, the only aspect of Šulgi's status as supremely just which cannot be reduced to a repetition of inherited commonplaces appears to be the metrological reform; even in Old Babylonian times, part of the pride of the scribes resided in their being (for most of them evidently only by proxy) the counsellors of kings who, on their part, were supposed to ensure affluence and justice, the latter at least in part identified with "just measure".

Much of this sounds fairly familiar. The modern state is only legitimate if its taxation is in agreement with rules, and since long the state guarantees trade and wealth first by providing (supposedly) stable currency and by taking care of a common metrology (most famous in the latter respect is the French Revolutionary introduction of the metric system, but this was not the beginning).

All of this, however, are functional requirements, and as in the case of the conceptualization of "mathematics" there may be a break in Mesopotamian culture in this respect with the Kassite take-over. The metrological innovations of the later second millennium were certainly not devoid of mathematical rationality, but the use of (normalized) seed measures indicates that mathematical regularity was no longer ideologically hegemonic; and the prestigious scholar-scribes who advised the Neo-Assyrian kings were concerned with omina and apotropaic ritual, not with justice, mathematical or otherwise. At this point of history, there was no longer anything to transmit.

And indeed, when Solon made metrological innovation part of his reform, he did so in a way which would have made any mathematically competent Mesopotamian scribe since Uruk IV laugh with contempt, "sixty-three minas [against formerly sixty] going to the talent; and the odd three minas were distributed among the staters and the other values".[28]

This apparent absence of a "meta-mathematical legacy" – which would be confirmed if we looked at other aspects of metamathematics – should be no surprise. After all, attitudes, knowledge and practices at this level are much more intimately bound up with culture as a whole than, say, a numerical

[28] Aristotle, *Constitution of Athens*, Ch. 10, trans. [Barnes 1984: II, 2346].

approximation to the ratio between a circular diameter and the perimeter. Even our belief to have inherited the metamathematics of the Hellenistic mathematicians is largely an illusion,[29] which we are only able to uphold by reading (that is, misreading) our own understanding for instance of what constitutes a proof into the Greek texts.

There certainly *is* a legacy from Mesopotamian mathematics in the modern world, as there was one in classical Antiquity; but there was no collective transmission of an organized whole understood as such. The most adequate metaphor is that of rich wreckage – similar to what Robinson Crusoe brought ashore from his ship. It did not allow him to reconstruct the European civilization from which it came, but it was essential by providing the tools for his construction of a somewhat civilized one-man world.

In other respects, Mesopotamian culture was certainly much more accessible as an integrated whole in the times of Solon, Alexander and Cicero; the reason that its mathematics was not was that Mesopotamian mathematics had been shipwrecked already when the Mycenaeans conquered the Minoans, a thousand years earlier.

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[29] [Netz 1998, 1999] offer full documentation.

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